А С Т Р О Ф И З И К А

ТОМ 68 ФЕВРАЛЬ, 2025 ВЫПУСК 1

DOI: 10.54503/0571-7132-2025.68.1-95

A COSMOLOGICAL MODEL WITH SECOND LAW OF THERMODYNAMICS IN f(R, T) GRAVITY

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Received 21 September 2024
Accepted 14 February 2025

In the context of $f(R, T)$ modified gravity theory, we investigate a cosmological model with homogeneous and anisotropic properties, specifically the Locally Rotationally Symmetric (LRS) Bianchi type-I model. By considering Einstein's field equations in $f(R, T)$ gravity, we solve them with the choice $f(R, T) = R + 2f(T)$, where R represents the Ricci scalar and T denotes the trace of the stress-energy momentum tensor T_{ij}. In this case, we set $f(T) = -\lambda T$, with λ being an arbitrary constant. It is worth noting that the cosmic jerk parameter j is directly proportional to the negative value of the deceleration parameter q, namely $j \propto -q$. We analyze the physical and geometrical properties of the models, and also employ the statefinder diagnostic pair to gain insight into the geometrical nature of the model. We also investigate the validity of the generalized second law of thermodynamics (GSLT) on the apparent and event horizons. Our findings reveal that GSLT holds on both the horizons.

Keywords: $f(R,T)$: *Bianchi type-I: Jerk parameter: generalised second law of thermodynamics: deceleration parameter: statefinder parameter*

1. *Introduction*. The discovery of the accelerating expansion of the Universe has been a significant advancement in modern cosmology [1-8]. This phenomenon is attributed to dark energy (DE), an exotic form of energy with negative pressure, which currently constitutes approximately 70% of the total energy content of the cosmos [9-11]. The cosmological constant Λ, characterized by the equation of state (EOS) $\omega = p/\rho$ where p represents the pressure and ρ is the energy density of DE with $\omega = -1$, is considered the most appealing and simplest candidate for DE. However, the cosmological constant faces challenges such as the fine-tuning problem and cosmic coincidence problem [12,13]. To address these issues, various dynamical scalar fields have been proposed as alternatives to DE, including quintessence [9-11,14-16], k-essence [17,18], phantom [19] and quintom fields [20,21].

On the other hand, modified gravity theory is the prominent gravity theory which can explain the present acceleration of the universe without any dark energy. It may also provide the explanation of dark matter. It may resolve the coincidence problem simply by the fact of the universe expansion, describe the transition from deceleration to acceleration of the universe and also useful for high-energy physics problems (i.e., unifications of all interactions, hierarchy problem resolution). Even if the current universe is entering the phantom phase, modified gravity effectively describes the transition from the non-phantom to phantom era without the need to introduce exotic matter (phantom) with extremely strange properties [22].

The modified gravity description of our universe cosmological evolution is one physically appealing theoretical framework, which can potentially explain the various evolution era's of the universe, for the simple reason that it can provide a unified and theoretically consistent description. In addition, modified gravity provides an alternative view of classical particle physics problems, like the baryogenesis issue. Particularly, it is possible to generate non-zero baryon to entropy ratio in the universe by using the gravitational baryogenesis mechanism [23]. Then, in the context of modified gravity it is possible to generalize the gravitational baryogenesis mechanism, and various proposals towards this issue have appeared in the literature [24].

The $f(R,T)$ gravity theory, proposed by Harko et al. [25], is an intriguing and promising version of modified gravity. It introduces a gravitational Lagrangian that is an arbitrary function of the Ricci scalar R and the trace of the stress-energy tensor T. In their work, Harko et al. derived the gravitational field equations in the metric formalism and the equation of motion for test particles, which arises from the covariant divergence of the stress-energy tensor. These $f(R,T)$ gravity models offer an explanation for the cosmic accelerated expansion observed in the late Universe.

Several researchers have since investigated cosmological models in $f(R,T)$ gravity within different Bianchi-type space-times. Specifically, Chaubey and Shukla [26], Adhav [27], Samanta [28], and Reddy et al. [29-31] have studied such models. Tiwari et al. [32] found an exact solution for the field equations of $f(R,T)$ gravity in the LRS Bianchi type-I model, assuming a linear relationship between the deceleration parameter and the Hubble parameter. Sofuoğlu [33] reconstructed the $f(R,T)$ model, allowing for the Gödel Universe. Tiwari et al. [34] investigated the time dependence of the gravitational and cosmological constants by considering a Bianchi type-I universe in $f(R,T)$ gravity. Tiwari and Beesham [35] examined the LRS Bianchi type-I space-time with a decaying cosmological term in this theory. Tiwari et al. [36] studied the Bianchi type-I space-time with a constant jerk parameter $j=1$ in $f(R,T)$ gravity. Chaubey and Shukla [37] explored the exact solutions for anisotropic Bianchi cosmological models in $f(R,T)$ gravity with a time-dependent cosmological constant $\Lambda(t)$. Singh and Bishi [38] discussed the presence of a cosmological constant Λ and a quadratic EOS in Bianchi type-I Universe within $f(R,T)$ gravity. Bharali and

Das [39] investigated the Bianchi type VI_0 space-time with modified Renyi holographic dark energy (MRHDE) in $f(R,T)$ gravity. Kumrah et al. [40] explored a homogeneous and isotropic cosmological model within the framework of $f(R,T)$ gravity, where the gravitational and cosmological constants are generalized as coupling scalars. Mishra et al. [41] presented a Bianchi type-I metric with an anisotropic variable parameter in $f(R,T)$ gravity. Nagpal et al. [42] have studied flat FLRW Universe in $f(R,T)=R+\alpha R^2+2\lambda T$ gravity with α being an arbitrary constant.

In recent years, Bianchi Universes have gained significance in observational cosmology due to the findings from the WMAP data [43-45]. These data suggest the need for an extension to the standard cosmological model, incorporating a positive cosmological constant that exhibits similarities with the Bianchi morphology [46-51]. Various studies have explored the implications of varying vacuum energy density in this context [52-62].

Interestingly, contrary to generic inflationary models [63-69], the WMAP data suggest that the Universe should possess a slightly anisotropic spatial geometry even after the inflationary phase. This indicates a non-trivial isotropization history of the Universe influenced by the presence of an anisotropic energy source. To account for the observed homogeneity and flatness of the Universe, it is commonly assumed that the Universe underwent a period of exponential expansion [63,65-67]. The majority of discussions about the expansion of the Universe take place within the framework of the homogeneous and isotropic Friedman-Robertson-Walker (FRW) cosmology. This preference is primarily due to the simplicity of the field equations and the availability of analytical solutions in most cases. However, there is no compelling physical reason to assume homogeneity prior to the inflationary period. Although dropping the homogeneity assumption would result in an intractable problem, relaxing the assumption of isotropy can lead to anisotropy. Several authors [70-75] have studied specific cases of anisotropic models and found that the predictions of the FRW model remain largely unaffected even when significant anisotropies were present before the inflationary period.

Furthermore, gravitational thermodynamics plays a crucial role in determining the viability of cosmological models. If two cosmological models satisfy the same observational constraints but one adheres to thermodynamic laws while the other does not, the later can be ruled out. Therefore, it is essential for any physical system to comply with thermodynamic laws. In this regard, extensive research has been done on the apparent and event horizons within various gravity theories [76-79]. The Generalized Second Law of Thermodynamics (GSLT) has garnered significant interest in the context of an accelerating Universe. Wang et al. demonstrated that thermodynamic laws are satisfied on the apparent horizon but fail to hold on the event horizon [80].

In [81], the second law of thermodynamics was discussed in the context of horizon cosmology. They consider various forms of entropy (i.e., Tsallis entropy, Renyi entropy, Kaniadakis entropy etc.) on the apparent horizon and determine the appropriate condition for entropic parameters for validation of the second law of thermodynamics. They found that the second law of thermodynamics is satisfied during wide range of cosmic eras of the universe particularly, from inflation to radiation-dominated eras followed by the reheating stage.

Moreover, in another paper [82], authors have discussed various issues that arise in the relationship of gravity and thermodynamics, where thermodynamic law is given by $TdS = -dE + WdV$. Also, they discussed the different problems that lead to some inconsistency in the Equation of State (EoS) parameter. They modified the thermodynamic law to $TdS = -dE + \rho dV$ on the apparent horizon to get rid of this issue and found that the modified thermodynamic law is valid for all values of EoS.

However, Chakraborty later showed that by modifying the horizon temperature, the GSLT can be satisfied on the event horizon [83]. Consequently, numerous studies have been undertaken to investigate the validity of the GSLT in the context of the event horizon [84-88]. Moreover, the validity of the GSLT has been explored in the framework of anisotropic Bianchi-I Universe models. Sharif and Saleem demonstrated that the GSLT is satisfied on the apparent horizon in the Bianchi-I model [89]. Their findings reveal that the GSLT consistently holds on the apparent horizon. In a separate study, Sharif and Khanum investigated the validity of the GSLT, considering various parameters such as shear, skewness, and equation of state in an anisotropic dark energy model [90].

This paper focuses on investigating the LRS Bianchi type-I cosmology within the framework of the modified $f(R,T)$ gravity theory. Specifically, we consider the choice $f(R,T) = R + 2f(T)$, where $f(T) = -\lambda T$, λ is an arbitrary constant. By utilizing this specific form, we obtain explicit solutions for the field equations, which are discussed in detail in Section 4. To provide a comprehensive understanding, we first introduce the basic formalism of $f(R,T)$ gravity in Section 2. The field equations are then presented in Section 3. Moving forward, Section 5 is devoted to examining the GSLT (Generalized Second Law of Thermodynamics) on both the apparent and event horizons. Furthermore, we explore the statefinder diagnostic, the physical acceptability of the solutions, and engage in graphical discussions of various parameters in Sections 6, 7, and 8 respectively. Finally, the paper ends with concluding remarks in Section 9.

2. *The basic formalism of f(R, T) gravity*. The gravitational action of $f(R,T)$ gravity is given by [25]

$$S = -\frac{1}{16\pi G c^4}\int f(R,T)\sqrt{-g}\, d^4x + \int L_m \sqrt{-g}\, d^4x. \qquad (1)$$

where $f(R,T)$ is an arbitrary function of the ricci scalar R and the trace T of the energy-momentum tensor $T_{\mu\nu}$ i. e. ($T = g^{\mu\nu}T_{\mu\nu}$), and L_m corresponds to the matter Lagrangian density and g is the determinant of metric tensor $g_{\mu\nu}$.

Using natural units ($c = 1 = 8\pi G$), a variation of action of Eq. (1) w.r. to metric tensor gives the following field equations of $f(R,T)$ gravity

$$f_R(R,T)R_{\mu\nu} - \frac{1}{2}f(R,T)g_{\mu\nu} - \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)f_R(R,T) = \\ -T_{\mu\nu} - f_T(R,T)T_{\mu\nu} - f_T(R,T)\Theta_{\mu\nu}\,, \tag{2}$$

where $f_R = \partial f(R,T)/\partial R$, $f_T = \partial f(R,T)/\partial T$, $\Box = \nabla^\mu\nabla_\mu$ is the D'Alembert operator, ∇_μ is the covariant derivative, $R_{\mu\nu}$ is the Ricci tensor, and $T_{\mu\nu}$ is the energy-momentum tensor given by

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\,L_m\right)}{\delta g^{\mu\nu}}, \tag{3}$$

and $\Theta_{\mu\nu}$ is

$$\Theta_{\mu\nu} = g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}. \tag{4}$$

Using Eqs. (3) and (4), we obtain

$$\Theta_{\mu\nu} = -2T_{\mu\nu} + g_{\mu\nu}L_m - 2g^{\alpha\beta}\frac{\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}. \tag{5}$$

By contracting Eq. (2), we get

$$f_R(R,T)R + 3f_R(R,T) - 2f(R,T) = \{-1 - f_T(R,T)\}T - f_T(R,T)\Theta. \tag{6}$$

where $\Theta = g^{\mu\nu}\Theta_{\mu\nu}$. If we assume that the matter Lagrangian density L_m depends on the metric tensor components $g_{\mu\nu}$ and does not depend on its derivatives, then Eq. (3) reads

$$T_{\mu\nu} = g_{\mu\nu}L_m - 2\frac{\partial L_m}{\partial g^{\mu\nu}}. \tag{7}$$

If the matter-energy source of the Universe is a perfect fluid, then the energymomentum tensor can be defined as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}\,, \tag{8}$$

where ρ and p are the energy density and the pressure of the fluid, respectively, and u^μ is the four-velocity vector satisfying $u^\mu u_\mu = -1$ and $u^\nu\nabla_\mu u_\nu = 0$. Now, for a perfect fluid distribution one can write the matter Lagrangian density as $L_m = -p$, which on using, Eq. (5) gives

$$\Theta_{\mu\nu} = -pg_{\mu\nu} - 2T_{\mu\nu}\,. \tag{9}$$

Then the field equations (2) take the form

$$f_R(R,T)R_{\mu\nu}-\frac{1}{2}f(R,T)g_{\mu\nu}-\left(\nabla_\mu\nabla_\nu-g_{\mu\nu}\Box\right)f_R(R,T)= \\ -T_{\mu\nu}+f_T(R,T)\left(T_{\mu\nu}+pg_{\mu\nu}\right). \tag{10}$$

We note that Harko et al. [25] have mentioned the following functional forms of $f(R,T)$ function:

$$f(R,T)=\begin{cases} R+2f(T) \\ f_1(R)+f_2(T) \\ f_1(R)+f_2(R)f_3(T). \end{cases} \tag{11}$$

In this paper, we focus on the first one of these functional forms i.e. $f(R,T)= =R+2f(T)$ and choose $f(T)=-\lambda T$, where λ is an arbitrary constant. For this choice of the function, Eq. (10) becomes

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=-(1+2\lambda)T_{\mu\nu}+\lambda(-T-2p)g_{\mu\nu}. \tag{12}$$

A comparison of Eq. (12) with the following Einstein's field equations

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=-T_{\mu\nu}+\Lambda g_{\mu\nu}, \tag{13}$$

yields $\Lambda=\Lambda(T)=-\lambda(T+2p)$. Thus, one can write the field equations of $f(R,T)$ gravity with varying cosmological constant Λ as

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=-(1+2\lambda)T_{\mu\nu}+\Lambda g_{\mu\nu}. \tag{14}$$

3. *Line element and field equations*. The spatially homogeneous and anisotropic LRS Bianchi type-I Universe model is described by the line element

$$ds^2=-dt^2+A^2dx^2+B^2\left(dy^2+dz^2\right), \tag{15}$$

where A and B are time-dependent metric potentials. For the model defined by the line element (15), the field equations (14) in $f(R,T)$ gravity give the following system of equations

$$2\frac{\ddot{B}}{B}+\frac{\dot{B}^2}{B^2}=\lambda\rho-(1+7\lambda)p, \tag{16}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}=\lambda\rho-(1+7\lambda)p, \tag{17}$$

$$2\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}^2}{B^2}=(1+3\lambda)\rho-5\lambda p, \tag{18}$$

where the dot (.) represent time derivative. Using the expression of the trace of the energy-momentum tensors $T=-\rho+3p$, yields $\Lambda=\lambda(\rho-5p)$.

The spatial volume V, mean scale factor a and the mean Hubble parameter

H for the Bianchi type-I Universe are given by

$$V = AB^2 , \tag{19}$$

$$a = \left(AB^2\right)^{1/3} = V^{1/3} , \tag{20}$$

$$H = \frac{1}{3}\left(H_x + H_y + H_z\right), \tag{21}$$

where H_x , H_y and H_z are directional Hubble parameters in the directions of x; y and z , respectively, which are defined as

$$H_x = \frac{\dot{A}}{A}, \quad H_y = H_z = \frac{\dot{B}}{B}. \tag{22}$$

Eqs. (21) and (22) provide us an important relation:

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(H_x + 2H_y\right). \tag{23}$$

The expansion rate θ and shear scalar σ are obtained as follows

$$\theta = u^{\mu}_{;\mu} = 3\frac{\dot{a}}{a}, \tag{24}$$

$$\sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} = \frac{k^2}{a^6}. \tag{25}$$

where $\sigma_{\mu\nu}$ is the shear tensor and k is a constant which comes from the anisotropy of the model. For LRS Bianchi type-I model, the average anisotropy parameter A_p and deceleration parameter q are defined as

$$A_p = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 . \tag{26}$$

$$q = \frac{-a\ddot{a}}{\dot{a}^2} = -\left(\frac{\dot{H} - H^2}{H^2}\right). \tag{27}$$

Thus, field equations (16)-(18), can be written in terms of Hubble and deceleration parameters as

$$3H^2 = \left(1 + 2\lambda\right)\rho + \sigma^2 + \Lambda . \tag{28}$$

$$H^2\left(2q - 1\right) = \left(1 + 2\lambda\right)p + \sigma^2 - \Lambda . \tag{29}$$

One can express Eq. (28) in the form of

$$\frac{\sigma^2}{3H^2} = 1 - \frac{\left(1 + 2\lambda\right)\rho}{3H^2} - \frac{\Lambda}{3H^2} = 1 - \Omega - \Omega_\Lambda , \tag{30}$$

where $\Omega_\Lambda = \rho_\Lambda / \rho_c$ is cosmological constant density parameter and $\Omega = \Omega_m + \Omega_\lambda$ is total density parameter. Here $\rho_c = 3H^2$ is critical density, $\rho_\Lambda = \Lambda$ is cosmological constant density, $\Omega_m = \rho / \rho_c$ is density parameter of matter and $\Omega_\lambda = \rho_\lambda / \rho_c$

with $\rho_\lambda = 2\lambda\rho$ may be considered as a correction term to density parameter of matter which comes from $f(R,T)$.

From Eqs. (16) and (17), we have

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{k_1}{a^3}, \tag{31}$$

$$A = Bk_2 \exp\left(\int \frac{k_1}{a^3} dt \right), \tag{32}$$

where k_1 and k_2 are constants of integration.

Using Eqs. (20) and (32), we get the scale factors A and B as

$$A = ak_2^{2/3} \exp\left(\int \frac{2k_1}{3a^3} dt \right), \tag{33}$$

$$B = ak_2^{-1/3} \exp\left(-\int \frac{k_1}{3a^3} dt \right). \tag{34}$$

4. *Solution of the field equations*. Eqs. (16)-(18) form a system of three independent equations involving four unknowns: A, B, ρ and p. To fully solve this system, we need to make one physically reasonable assumption. Therefore, we adopt a kinematical condition where the jerk parameter j is directly proportional to the negative of the deceleration parameter q i.e. $j \propto -q$. The jerk parameter represents the dimensionless third derivative of the average scale factor a w.r. to cosmic time t. This parameterization offers an alternative approach to describe a model that closely resembles the ΛCDM model [91].

In the flat ΛCDM models, the jerk parameter remains constant, specifically $j=1$ [92]. The jerk parameter, its implications, and further details can be found in the works of Tiwari et al. [36,93], Poplawski [94] and relevant references therein. For our study, we assume the proportionality $j \propto -q$, thus incorporating the relationship between the jerk and deceleration parameters.

$$\frac{\dddot{a}}{aH^3} + \beta q = 0, \tag{35}$$

where $j = \dddot{a}/aH^3$ and β is a constant of proportionality.

Without loss of generality, we take $\beta = 1$ and solving Eq. (35), we get

$$a = k_1 \sinh(k_2 t + k_3), \tag{36}$$

where k_2 and k_3 are constants of integration. For the above mean scale factor, the solutions of metric potentials are given in Eqs. (33) and (34) are

$$A = k_1 k_2^{2/3} \sinh(k_2 t + k_3) \exp\left(\frac{2}{3k_1^2} F(t) \right), \tag{37}$$

$$B = k_1 k_2^{-1/3} \sinh(k_2 t + k_3) \exp\left(-\frac{1}{3k_1^2} F(t) \right), \tag{38}$$

where

$$F(t) = \int [\sinh(k_2 t + k_3)]^{-3} \, dt$$
$$= 1 + \frac{2}{3}\cosh^2(k_2 t + k_3) + \frac{3}{5}\cosh^2(k_2 t + k_3) + o[\cosh(k_2 t + k_3)]^6 \,. \tag{39}$$

For this model, the directional Hubble parameters H_x, H_y and H_z are obtained as

$$H_x = \frac{\dot{A}}{A} = k_2 \coth\tau + \frac{2}{3k_1^2 \sinh^3\tau}, \tag{40}$$

$$H_y = H_z = \frac{\dot{B}}{B} = k_2 \coth\tau - \frac{1}{3k_1^2 \sinh^3\tau}, \tag{41}$$

where $\tau = k_2 t + k_3$. The anisotropy parameter A_p is obtained as

$$A_p = \frac{2}{27 k_1^4 k_2^2 \coth^2\tau \sinh^6\tau}. \tag{42}$$

Anisotropy, in general, affects the dynamics of the universe. The anisotropy parameter A_p gives a measure of the anisotropy of the model and is given by Eq. (42), which is large early on as $t \to 0$ but decreases rapidly [95]. Hence, our model reaches to isotropy after some finite time which matches with the recent observations as the universe is isotropic at large scale [96]. For $\beta = 1$, the universe has an accelerated expansion throughout the evolution which reseambles with the result obtained in [97]. Thus as the universe evolves, the anisotropy damps out, leading to the currently observable universe [98].

Further, the Hubble parameter H, spatial volume V, expansion scalar θ, shear scalar σ^2 and deceleration parameter q take the following forms

$$H = k_2 \coth\tau, \tag{43}$$

$$V = k_1^3 \sinh^3\tau, \tag{44}$$

$$\theta = 3H = 3k_2 \coth\tau, \tag{45}$$

$$\sigma^2 = \frac{k^2}{k_1^6 \sinh^6\tau}, \tag{46}$$

$$q = -\tanh^2\tau. \tag{47}$$

Using Eqs. (16)-(18), (36), (40) and (41), energy density ρ and pressure p are obtained as

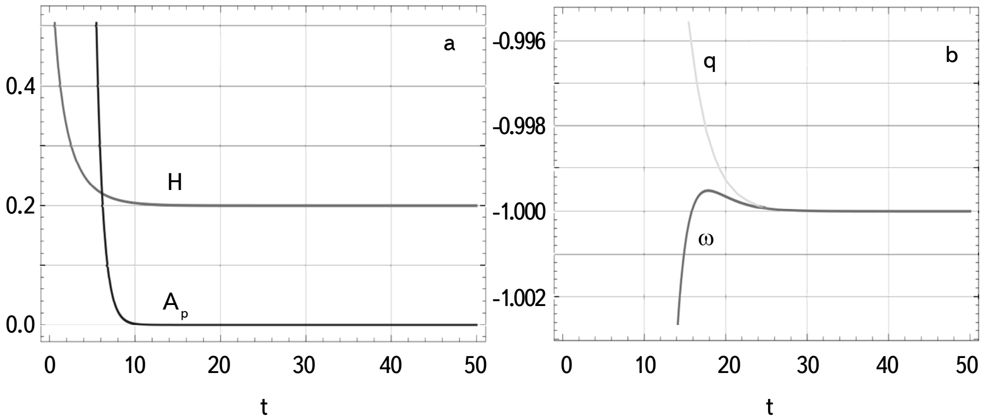

Fig.1. (a) H, A_p, (b) q, ω.

$$p = \left(\frac{3\lambda+1}{-16\lambda^2-10\lambda-1}\right)\left[3k_2^2\left(\frac{2\lambda+1}{3\lambda+1}\right)\coth^2\tau - 2k_2\,\mathrm{csch}^2\tau + \left(\frac{2-2k_2}{k_1^2}\right)\coth\tau\,\mathrm{csch}^3\tau + \frac{1}{3k_1^4\sinh^6\tau}\left(\frac{4\lambda+1}{3\lambda+1}\right)\right]. \quad (48)$$

$$\rho = \left[3k_2^2\coth^2\tau - \frac{1}{3k_1^4\sinh^6\tau} - \left(\frac{15\lambda^2+5\lambda}{16\lambda^2+10\lambda+1}\right)\right]\left[3k_2^2\left(\frac{2\lambda+1}{3\lambda+1}\right)\coth^2\tau - 2k_2\,\mathrm{csch}^2\tau + \left(\frac{2-2k_2}{k_1^2}\right)\coth\tau\,\mathrm{csch}^3\tau + \frac{1}{3k_1^4\sinh^6\tau}\left(\frac{4\lambda+1}{3\lambda+1}\right)\right]\left(\frac{1}{1+3\lambda}\right). \quad (49)$$

The EOS parameter $\omega = p/\rho$ can be obtained by dividing Eqs. (48) and (49). For the present model, we obtain, the density parameter $\Omega = \rho/3H^2$ as

$$\Omega = \frac{\rho}{3k_2^2\coth^2\tau}, \quad (50)$$

where ρ is given by Eq. (49). Fig.1 shows graphical representation of these results. In the following we shall discuss the GSLT on the apparent and event horizon in Bianchi-I model.

5. *Generalized second law of thermodynamics*. This section is devoted to study the generalised second law thermodynamics (GSLT). The GSLT is one of the most prominent principles to check the viability of a cosmological model. It states that the rate of change of the total entropy of the system must be non-negative, i.e. derivative (w.r.t. to cosmic time) sum of horizon entropy and entropy of the matter within the horizon is always greater than or equal to zero. To evaluate the rate of change of the entropy of the matter within the horizon, we

use Gibb's equation and for the horizon entropy, we use the first law of thermodynamics. In this study, we shall use Hawking and modified Hawking temperatures for the homogeneous and anisotropic Bianchi type-I Universe bounded by apparent and event horizon separately. While the apparent horizon forms a Bekenstein's system in the accelerating Universe, the event horizon does not exhibit the usual definitions of entropy and temperature as proposed by Bekenstein [80]. However, it has been demonstrated that the event horizon can be considered as a Bekenstein's system in the context of an accelerating Universe through modifications to the Hawking temperature [83]. Now from the first law of thermodynamics, we get

$$T_X dS_X = -dE_X = 4\pi R_X^3 H(\rho + p)dt, \tag{51}$$

where dE_X is the energy crossing through the horizon in time dt (here $X = A$ denote the apparent and $X = E$ denote event horizon). Also, T_X and R_X denote the temperature and radius of the horizon respectively. From the above equation, we get the rate of change of horizon entropy as

$$\frac{dS_X}{dt} = \frac{4\pi R_X^3 H}{T_X}(\rho + p). \tag{52}$$

The Gibb's equation is given by [80,99]

$$T_X dS_m = dE_m + pdV, \tag{53}$$

where $E_m = \rho V$ is the energy flow across the horizon containing matter and $V = 4/3\pi R_X^3$ is the volume with R_X is the horizon (apparent or event) radius. Also, we assume that the temperature of the matter is the same as the temperature of the horizon (i.e., $T_m = T_X$) by the local equilibrium hypothesis as the temperature difference is very small between matter fluid and the horizon at cosmological scales [99-102]. So, the rate of change of the matter entropy inside horizon dS_m/dt is given by

$$\frac{dS_m}{dt} = \frac{4\pi R_X^2}{T_X}(\rho + p)\left(\frac{dR_X}{dt} - HR_X\right). \tag{54}$$

Now, adding Eqs. (52) and (54), we get the total rate of change of entropy dS_{TX}/dt at the horizon as

$$\frac{dS_{TX}}{dt} = \frac{dS_X}{dt} + \frac{dS_m}{dt} = \frac{4\pi R_X^2}{T_X}(\rho + p)\frac{dR_X}{dt}. \tag{55}$$

For validity of GSLT, we must have the condition $dS_{TX}/dt \geq 0$. Assuming, a positive temperature, we see that GSLT will be valid as long as $(\rho + p) \geq 0$ and $dR_X/dt \geq 0$ (or $(\rho + p) \leq 0$ and $dR_X/dt \leq 0$). In what follows, we shall discuss the validity of GSLT for the homogeneous and anisotropic Bianchi type-I Universe bounded by apparent and event horizon respectively in the following subsections:

5.1. *Apparent horizon*. For the homogeneous and anisotropic Bianchi type-I Universe, the radius of the apparent horizon is inverse of the Hubble parameter and given by

$$R_A = \frac{1}{H} \tag{56}$$

and the rate of change of the apparent horizon is given by

$$\frac{dR_A}{dt} = -\frac{dH}{dt}\frac{1}{H^2} = 1 + q\,. \tag{57}$$

The Hawking temperature associated with the apparent horizon is [103]

$$T_A = \frac{1}{2\pi R_A} = \frac{H}{2\pi}\,. \tag{58}$$

Now, the rate of change of horizon entropy dS_A/dt and matter entropy dS_m/dt are given by Eqs. (52) and (54) as,

$$\frac{dS_A}{dt} = \frac{4\pi^2}{H^3}(\rho + p), \tag{59}$$

$$\frac{dS_m}{dt} = \frac{4\pi^2}{H^3}(\rho + p)\left(\frac{dR_A}{dt} - HR_A\right). \tag{60}$$

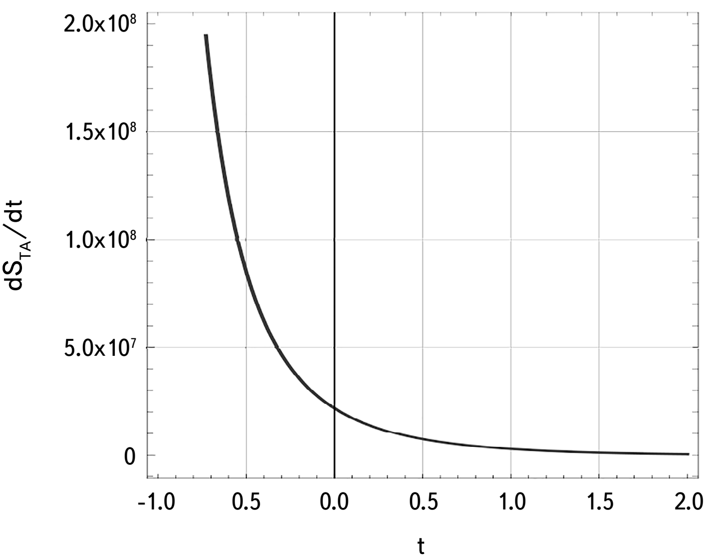

Fig.2. The rate of change of the total entropy at the apparent horizon dS_{TA}/dt is plotted against time (t) with $k_2 = 0.2$, $k_3 = 0.3$ and $\lambda = -0.1$.

Therefore, the rate of change of total entropy at the apparent horizon from Eq. (55) is given by

$$\frac{dS_{TA}}{dt}=\frac{4\pi^2}{H^3}(\rho+p)\frac{dR_A}{dt}. \tag{61}$$

Now using Eqs. (28), (29) and (57), the above Eq. (61) becomes

$$\frac{dS_{TA}}{dt}=\frac{8\pi^2\left\{H^2(1+q)^2-\sigma^2(1+q)\right\}}{H^3(1+2\lambda)}. \tag{62}$$

Eq. (62) represents the rate of change of total entropy on the apparent horizon. The validity of the GSLT on the apparent horizon requires that $dS_{TA}/dt\geq 0$. Due to the complexity of the expression, we examine the validity of GSLT through a graphical approach. Fig.2 illustrates the plot of the rate of change of total entropy dS_{TA}/dt against cosmic time *t*. It is evident from the graph that GSLT is consistently satisfied on the apparent horizon.

5.2. *Event horizon*. The radius of event horizon R_E is given by

$$R_E=a(t)\int_t^\infty\frac{dt'}{a(t')}. \tag{63}$$

From Eq. (63) we get,

$$\frac{dR_E}{dt}=HR_E-1. \tag{64}$$

In this case, the thermodynamical system bounded by the event horizon may not be a Bekenstein system [80], so we consider modified Hawking temperature instead of Hawking temperature. The modified Hawking temperature on the event horizon is defined as [83]

$$T_E=\frac{H^2R_E}{2\pi}. \tag{65}$$

Now, the rate of change of horizon entropy $\dot{S}_E$ and matter entropy $\dot{S}_m$ are respectively given by the Eqs. (52) and (54) as,

$$\frac{dS_E}{dt}=\frac{8\pi^2R_E^2}{H}(\rho+p), \tag{66}$$

$$\frac{dS_m}{dt}=\frac{8\pi^2R_E}{H^2}(\rho+p)\left(\frac{dR_E}{dt}-HR_E\right). \tag{67}$$

Therefore, the rate of change of total entropy at the event horizon from Eq. (55) is given by

$$\frac{dS_{TE}}{dt}=\frac{8\pi^2R_E^2}{H^2}(\rho+p)\left(\frac{1}{R_A}-\frac{1}{R_E}\right). \tag{68}$$

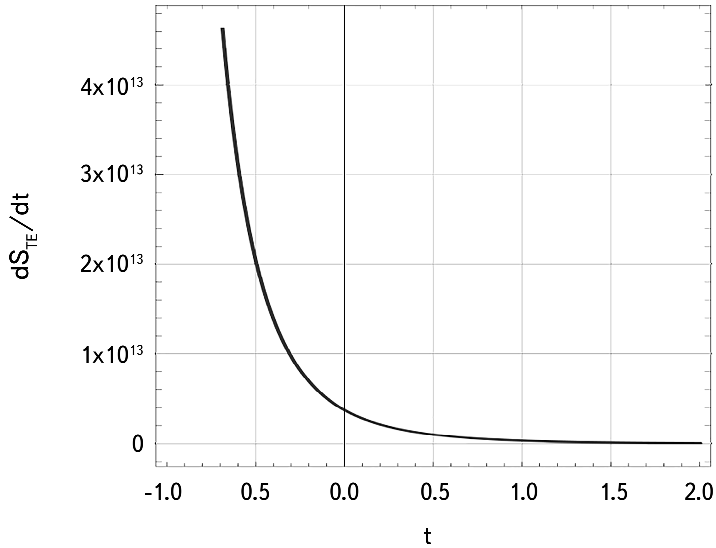

Fig.3. The rate of change of the total entropy at the event horizon dS_{TE}/dt is plotted against time t with $k_2 = 0.2$, $k_3 = 0.3$ and $\lambda = -0.1$.

From the above Eq. (68) we see that GSLT is satisfied as long as $(\rho + p) \geq 0$ under the realistic assumption that the radius of event horizon is greater than the radius of apparent horizon [83]. Using Eqs. (28), (29) and (64), the above Eq. (68) becomes

$$\frac{dS_{TE}}{dt} = \frac{16\pi^2 R_E^2 \left\{H^2(1+q) - \sigma^2\right\}}{H^2(1+2\lambda)} \left(\frac{1}{R_A} - \frac{1}{R_E}\right). \tag{69}$$

In order to ensure the validity of GSLT at the event horizon, it is necessary to have $dS_{TE}/dt \geq 0$. However, the expression of GSLT at the event horizon is quite complex. Therefore, we discuss it graphically. We have created a graphical representation that illustrates the plot of the rate of change of total entropy dS_{TE}/dt versus cosmic time t, as depicted in Fig.3. Upon observing the figure, it becomes evident that GSLT is consistently satisfied.

6. *Statefinder diagnostic*. Sahni et al. [104] introduced a statefinder diagnostic approach, which utilizes the third derivative of the average scale factor w.r. to cosmic time t, to define a geometrical statefinder pair $\{r, s\}$. This diagnostic tool serves as a useful test for distinguishing between different dark energy models. Moreover, the statefinder diagnostic pair also characterizes the ΛCDM model, where the cosmological constant Λ plays the role of dark energy.

The ΛCDM model is considered the fundamental model in the study of the

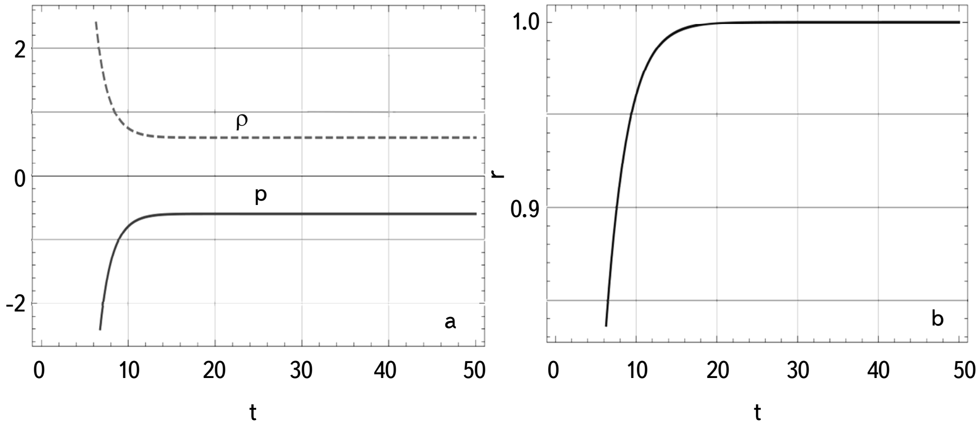

Fig.4. (a) p, ρ. (b) r.

evolution of the accelerating Universe, and it is characterized by the fixed point $\{r, s\} = \{1, 0\}$.

The state finder diagnostic pair is mathematically defined as follows:

$$r = 1 + \frac{3\dot{H}}{H^2} + \frac{\ddot{H}}{H^3}, \tag{70}$$

$$s = \frac{r-1}{3(q-1/2)}. \tag{71}$$

Now, we apply the statefinder diagnostic approach to our model for testing its behavior in accordance with ΛCDM model (Fig.4). For our model, the expression of parameters $\{r, s\}$ are obtained as follows

$$r = 1 - \text{sech}^2\tau, \tag{72}$$

$$s = \frac{2\text{sech}^2\tau}{3\left(1 + 2\tanh^2\tau\right)}. \tag{73}$$

7. *Physical acceptability of the solutions*. For the stability of corresponding solutions, we should check that our model is physically acceptable.

· Sound speed: It is required that the velocity of sound υ_s should be less than the velocity of light c. The positive value of υ_s implies that the model is stable whereas the negative value implies that the model is unstable.

The sound speed υ_s for our model is obtained as

$$\upsilon_s = \frac{dp}{d\rho} = \frac{l(t)}{m(t)},$$

where

$$l(t)=\left(\frac{3\lambda+1}{16\lambda^2+10\lambda+1}\right)\left[\left(6k_2^3\left(\frac{2\lambda+1}{3\lambda+1}\right)-4k_2^2\right)\coth\tau\,\mathrm{csch}^2\tau+\frac{2k_2(1-k_2)}{k_1^2}\mathrm{csch}^3\tau\left(\mathrm{csch}^2\tau+3\coth^2\tau\right)+\frac{8\lambda+2}{3\lambda k_1^4+k_1^4}\mathrm{csch}^6\tau\coth\tau\right],$$

$$m(t)=\frac{1}{1+3\lambda}\left\{-6k_2^3\coth\tau\,\mathrm{csch}^2\tau-\frac{2k_2}{k_1^4}\coth\tau\,\mathrm{csch}^6\tau-\left(\frac{15\lambda^2+5\lambda}{16\lambda^2+10\lambda+1}\right)\left[\left(6k_2^3\left(\frac{2\lambda+1}{3\lambda+1}\right)-4k_2^2\right)\coth\tau\,\mathrm{csch}^2\tau+\frac{2k_2(1-k_2)}{k_1^2}\mathrm{csch}^3\tau\left(\mathrm{csch}^2\tau+3\coth^2\tau\right)+\frac{8\lambda+2}{3\lambda k_1^4+k_1^4}\mathrm{csch}^6\tau\coth\tau\right]\right\}.$$

Fig.5b depicts the plot of sound speed with cosmic time. We observe that $\upsilon_s>1$ throughout the evolution of the Universe.

· Energy conditions (EC): The weak energy conditions (WEC) and dominant energy conditions (DEC) are given by (i) $\rho\geq 0$, (ii) $\rho-p\geq 0$ and (iii) $\rho+p\geq 0$. The strong energy conditions (SEC) are given by $\rho+3p\geq 0$ [105-108]. Various

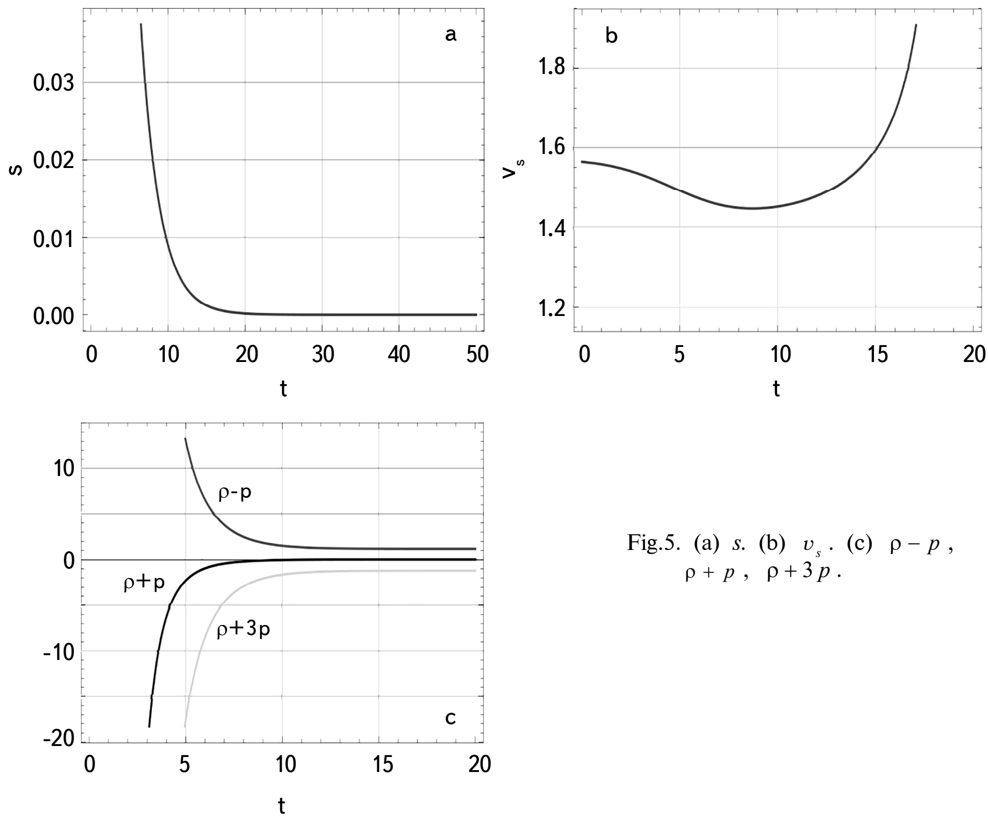

Fig.5. (a) s. (b) υ_s. (c) $\rho-p$, $\rho+p$, $\rho+3p$.

authors [109-112] have studied energy conditions in different theories of gravities. The lefthand side of energy conditions have been graphed in Fig.4a and Fig.5c. From these figures, we observe that WEC and DEC for the derived model are satisfied whereas SEC is violated.

8. *Graphical discussions*. In all the graphs, t denotes cosmic evolution time, generally measured in Giga years ($1\,\mathrm{Gyr} = 10^9$) years along the x axis. Along the y axis, all physical quantities like anisotropy parameter A_p, EOS parameter ω, energy conditions etc. are measured in geometrized units, where the speed of light $c = 1$ and the gravitational constant $G = 1$. The numerical values used in the graphs are $\lambda = -0.1$, $k_1 = 0.5$, $k_2 = 0.2$ and $k_3 = 0.3$.

9. *Conclusions*. In this study, we have investigated the properties of a spatially homogeneous and anisotropic LRS Bianchi type-I model within the framework of $f(R,T)$ gravity. Specifically, we consider the choice $f(R,T) = R + 2f(T)$, where $f(T) = -\lambda T$ and λ is an arbitrary constant. To fully solve the field equations, we adopt the condition $\dddot{a}/aH^3 + \beta q = 0$, where the jerk parameter j is directly proportional to the negative of the deceleration parameter q. As the cosmic time evolves, both the Hubble parameter H and the anisotropy parameter A_p decreases and eventually approaches to zero at the later stage of the Universe. This implies that the Universe exhibits anisotropy in its early stage and tends towards isotropy at later times. The deceleration parameter $q \to -1$, indicates that the model is experiencing cosmic acceleration. At the early stage of the Universe, the EOS parameter $\omega < -1$, suggests a behavior similar to phantom dark energy. However, as the Universe evolves, it approaches the phantom-divide line $\omega = -1$. Verma et al. [113] have arrived at a conclusion that EOS greater than -1 which is in line with the recent findings from DESI colloboration. The satefinder parameters $r \to 1$, $s \to 0$, respectively, as cosmic time progresses, indicating that our model corresponds to the ΛCDM model at the later epoch. Further, we check the validity of this model by examining the GSLT on both the apparent and event horizons. For this, we consider the Hawking temperature for the apparent horizon and the modified Hawking temperature for the event horizon. It is crucial for a viable model to satisfy both observational constraints and thermodynamic principles. Our investigation reveals that the GSLT is consistently fulfilled on both horizons. Therefore, based on thermodynamic considerations, it can be concluded that the given cosmological model is viable.

Throughout the evolution of the Universe, the sound speed υ_s remains positive, indicating the stability of our model. For our model, WEC is satisfied whereas DEC and SEC are violated. Thus, the derived solutions represent accelerating Universe models that are consistent with the current observations of SNe Ia and CMB.

Acknowledgments. The authors are grateful to their respective institutions for providing facilities and support to carry out this research work. The authors are also thankful to the anonymous reviewer for making valuable comments and suggestions which helped us to improve the quality of paper in present form.

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КОСМОЛОГИЧЕСКАЯ МОДЕЛЬ СО ВТОРЫМ ЗАКОНОМ ТЕРМОДИНАМИКИ В ГРАВИТАЦИИ f(R, T)

Р.К.ТИВАРИ[1], Д.БХАРАЛИ[2], Б.ЧЕТРИ[3], А.БИШАМ[4]

В рамках модифицированной теории гравитации $f(R,T)$ исследована космологическая модель с однородными и анизотропными свойствами, в частности, локально вращательно-симметричная (LRS) модель Бьянки типа I. Представлены решения уравнения поля Эйнштейна в гравитации $f(R,T) = \; = R + 2f(T)$, где R скаляр Риччи, а T - след тензора энергии импульса T_{ij}. В этом случае принято $f(T) = -\lambda T$, где λ произвольная константа. Надо отметить, что параметр космического толчка j прямо пропорционален отрицательному значению параметра замедления q, а именно $j \propto -q$. Анализированы физические и геометрические свойства моделей, использованы диагностические диаграммы, чтобы получить представление о геометрической природе модели. Рассмотрен вопрос применимости обобщенного второго закона термодинамики (GSLT) на видимом горизонте и горизонте событий. Полученные результаты показывают, что GSLT выполняется на обоих горизонтах.

Ключевые слова: $f(R,T)$: *тип Бьянки-I: параметр рывка: обобщенный второй закон термодинамики: параметр замедления: параметр определения состояния*

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